

Synchronization in abstract mean field models

W. Oukil,

Laboratory of Dynamical Systems (LSD), Faculty of Mathematics,
University of Sciences and Technology Houari Boumediene,
BP 32 El Alia 16111, Bab Ezzouar, Algiers, Algeria.

March 23, 2017

Abstract

We show in this paper a sufficient condition for the existence of solution, the synchronized and the periodic locked state in abstract mean field models or interconnected systems. This condition is true for a small perturbation independently of the number of oscillators. We show in addition a numerical example of linear mean field system.

Keywords: Coupled oscillators, abstract mean field models, interconnected systems, synchronization, desynchronization, periodic orbit.

1 Introduction

This article is a generalization of the result obtained in [8]. The class of abstract mean field systems that we study in this article is given by the two next systems. The *periodic not-perturbed* system

$$\dot{x}_i = F(X, x_i), \quad i = 1, \dots, N, \quad t \geq t_0, \quad (\text{PNP})$$

and the *perturbed* system

$$\dot{x}_i = F(X, x_i) + H_i(X), \quad i = 1, \dots, N, \quad t \geq t_0, \quad (\text{P})$$

where $N \geq 2$ and $X = (x_1, \dots, x_N)$ is the state of the system. $F : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ and $H = (H_1, \dots, H_N) : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are C^1 functions. We note Φ^t the flow of the system (P) (in particular of the system (PNP)). We have separated between periodic not-perturbed and perturbed system because the results seem not trivial for the periodic not-perturbed system. We prove in this article the existence of solution in one hand and in the other hand the existence of synchronized and periodic locked solution.

1.1 Notations and definitions

In this section, we introduce some notations and definitions. For $q, p \in \mathbb{N}^*$ let G be a function from \mathbb{R}^q to \mathbb{R}^p . Put $G = (G_1, \dots, G_p)$ we consider the quasi-norm on the space of continues functions from \mathbb{R}^q to \mathbb{R}^p defined by the next quantity

$$\|G\|_B = \sup_{Y \in B} \max_{1 \leq i \leq p} |G_i(Y)|,$$

$$\text{where } B = \{Y = (y_1, \dots, y_q) \in \mathbb{R}^q : \max |y_i - y_j| \leq 1\}.$$

This quasi-norm is a norm on the space of continues functions from B to \mathbb{R}^p . We note $d^i G$, $i = 1, 2, \dots$, the i^{th} differential of G . We define

$$\|dG\|_B = \max_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \|\partial_j G_i(Y)\|_B, \quad \|d^2 G\|_B = \max_{\substack{1 \leq i \leq p \\ 1 \leq j, k \leq q}} \|\partial_k \partial_j G_i(Y)\|_B.$$

Let $G : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$, $Y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and $z \in \mathbb{R}$, We note

$$\partial_i G(Y, z) = \begin{cases} \frac{\partial}{\partial z} G(Y, z), & i = N + 1 \\ \frac{\partial}{\partial y_i} G(Y, z), & i := 1, \dots, N. \end{cases}$$

A function $G : \mathbb{R}^q \rightarrow \mathbb{R}^p$ is called $\mathbb{1}$ -periodic in the sense of the following definition

Definition 1. [$\mathbb{1}$ -periodic function] Let $G : \mathbb{R}^q \rightarrow \mathbb{R}^p$ be a function and note $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^q$. The function G is called $\mathbb{1}$ -periodic if

$$G(Y + \mathbb{1}) = G(Y), \quad \forall Y \in \mathbb{R}^q.$$

Remark that the previous definition do not imply that the function G is periodic relative to each variable. This allows us to get a large class of mean field systems as a linear system given in a numerical example in the Section 6. Now we define a positive Φ^t -invariant set,

Definition 2. Suppose that the flow Φ^t of system (P) exists for every $t \geq t_0$. We say that a open set $C \subset \mathbb{R}^N$ is a positive Φ^t -invariant if $\Phi^t(C) \subset C$ for all $t \geq t_0$.

Synchronization and locking may have several meanings or definitions depending on the authors. We choose the following definitions.

Definition 3 (Dynamical oscillator). The oscillator $x_i(t)$ of a solution $X(t) = (x_1(t), \dots, x_N(t))$ of system (P) is called *dynamical* if there exists $t_0 \in \mathbb{R}$ such that

$$\inf_{t \geq t_0} \dot{x}_i(t) > 0.$$

Definition 4 (Synchronisation). We say that the oscillators $\{x_i(t)\}_{i=1}^N$ are synchronized if they are dynamical and if $\sup_{1 \leq i, j \leq N} |x_i(t) - x_j(t)|$ is bounded from above uniformly in time $t \geq t_0$.

Definition 5 (Periodic locked solution). We say that the oscillators $\{x_i(t)\}_{i=1}^N$ are periodically locked to the frequency $\rho > 0$ if they are synchronized and if there exist a periodic functions $\Psi_i(t)$ such that

$$x_i(t) = \rho t + \Psi_i(t), \quad \forall i = 1 \dots N, \quad \forall t \geq t_0.$$

1.2 Synchronization Hypothesis (H) et (H_*)

We consider tow hypotheses (H) and (H_*),

$$\begin{aligned} (H) \quad & \left\{ \begin{array}{l} F \text{ is } C^2, \text{ and } \max\{\|F\|_B, \|dF\|_B, \|d^2F\|_B\} < +\infty, \\ F \text{ is } \mathbb{1}\text{-periodic and } \min_{s \in [0,1]} F(s\mathbb{1}, s) > 0, \end{array} \right. \\ (H_*) \quad & \int_0^1 \frac{\partial_{N+1} F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} ds < 0. \end{aligned}$$

We call the hypothesis (H_*) the *synchronization hypothesis* provided that hypothesis (H) is satisfied. The hypothesis (H_*) comes from the fact that when $H \approx 0$ and $x_i \approx x_j (\approx x)$ the system (P) is equivalent to

$$\frac{d}{dt} x_i \approx F(x\mathbb{1}, x), \text{ and } \frac{d}{dt} (x_i - x_j) \approx \partial_{N+1} F(x\mathbb{1}, x) (x_i - x_j).$$

The condition $\min_{s \in [0,1]} F(s\mathbb{1}, s) > 0$ allow to have a dynamical oscillators as defined in definition 3.

1.3 Main Results

The following main result **I** shows the existence of the solution and a synchronized solution in the sense of definition 4

Main Result (I). *We consider the system (P). Suppose that F satisfies the hypotheses (H) and (H_*) then there exists $D_* > 0$ such that for all $D \in (0, D_*]$ there exists $r > 0$ and a open set C_r of the form,*

$$C_r := \left\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N : \exists \nu \in \mathbb{R}, \quad \max_i |x_i - \nu| < \Delta_r(\nu) \right\},$$

where $\Delta_r : \mathbb{R} \rightarrow (0, D]$ is a C^1 and 1-periodic function, such that for every C^1 function H satisfying $\|H\|_B < r$ we have

1. Existence of solution. The flow Φ^t of the system (P) exists for all initial condition $X \in C_r$ and for all $t \geq t_0$.
2. Synchronization. The open set C_r is positive Φ^t -invariant. Further, for every $X \in C_r$ we have

$$\min_{1 \leq i \leq N} \inf_{t \geq t_0} \frac{d}{dt} \Phi_i^t(X) > 0 \text{ and } |\Phi_i^t(X) - \Phi_j^t(X)| < 2D, \forall 1 \leq i, j \leq N, \forall t \geq t_0.$$

The next main result **II** shows the existence of a periodic locked solution in the sense of definition 5

Main Result (II). We consider the system (P). Suppose that F satisfies the hypotheses (H) and (H_*) then there exists $D_* > 0$ such that for all $D \in (0, D_*]$ there exists $r > 0$ such that for every C^1 and 1-periodic function H satisfying $\|H\|_B < r$, there exists an open set C_r (same in main result (I)) and a initial condition $X_* \in C_r$ such that

$$\Phi_i^t(X_*) = \rho t + \Psi_{i,X_*}(t), \quad \forall i = 1, \dots, N, \quad \forall t \geq t_0,$$

where $\rho > 0$ and $\Psi_{i,X_*} : \mathbb{R} \rightarrow \mathbb{R}$ are a C^1 and $\frac{1}{\rho}$ -periodic functions.

Remark. The result **I** can be generalized to a function $H(t, X)$ which depend on time t .

1.4 Remarks and motivation

Our results can be applied to the model of coupled oscillators to study the synchronization of biological oscillators as the Winfree [13] and the Kuramoto model [11] as illustrated in the following example.

Example 6. [Winfree and Kuramoto Models] Winfree [13] proposed a model describing the synchronization of a population of organisms or *oscillators* that interact simultaneously. The Winfree model is also studied in [10, 6, 12, 5, 2, 7, 9]. Kuramoto model is a refined model of the Winfree model. The Kuramoto model is applied for example in the Neurosciences to study the synchronization of neurones in the brain [3, 4]. We call natural frequency, the frequency of each oscillator, as if it were isolated from the others. The explicit Winfree [1] and Kuramoto model are defined by the following equation respectively

$$\dot{x}_i = \omega_i + \text{Win}(X, x_i), \quad i = 1 \dots N, \quad t \geq t_0, \quad (\text{W})$$

$$\dot{x}_i = \omega_i + \text{Kur}(X, x_i), \quad i = 1 \dots N, \quad t \geq t_0, \quad (\text{K})$$

where for $(\omega, \kappa) \in \mathbb{R}_+^2$, $\text{Win}(Y, z) = \omega - \kappa \frac{1}{N} \sum_{j=1}^N [1 + \cos(y_j)] \sin(z)$ and $\text{Kur}(Y, z) = \omega - \kappa \frac{1}{N} \sum_{j=1}^N \sin(y_j - z)$ for all $Y = (y_1, \dots, y_N) \in \mathbb{R}^N$ and $z \in \mathbb{R}$. $X(t) = (x_1(t), \dots, x_N(t))$ is the state of the systems, and $x_i(t)$ is the phase of the i^{th} -oscillator. The parameter $\kappa \geq 0$ is the strong coupling; the vector $(\omega_1 + \omega, \dots, \omega_N + \omega) \in \mathbb{R}^N$ is the vector of the natural frequencies.

Proposition 7. *There exists an open set of parameters $(\kappa, \omega) \in \mathbb{R}_+^2$, such that the functions Win and Kur of the systems (W) and (K) respectively, satisfies both hypotheses (H) and (H*).*

Proof. The function Win is C^2 and $2\pi\mathbb{1}$ -periodic. Further

$$\min_{s \in [0, 2\pi]} \text{Win}(s\mathbb{1}, s) > 0 \iff \forall \omega > (1 + \cos(\frac{\pi}{3})) \sin(\frac{\pi}{3})\kappa, \quad \forall s \in [0, 2\pi].$$

For every $\omega > (1 + \cos(\frac{\pi}{3})) \sin(\frac{\pi}{3})\kappa$ we have

$$\begin{aligned} \int_0^{2\pi} \frac{\partial_{N+1} \text{Win}(s\mathbb{1}, s)}{\text{Win}(s\mathbb{1}, s)} ds &= - \int_0^{2\pi} \frac{\kappa [1 + \cos(s)] \cos(s)}{\omega - \kappa (1 + \cos(s)) \sin(s)} ds \\ &= - \int_0^{2\pi} \frac{\kappa \sin^2(s)}{\omega - \kappa (1 + \cos(s)) \sin(s)} ds < 0. \end{aligned}$$

Same for the Kuramoto model, we have Kur is $2\pi\mathbb{1}$ -periodic, and

$$\min_{s \in [0, 2\pi]} \text{Kur}(s\mathbb{1}, s) > 0, \quad \forall \omega > 0, \quad \forall s \in [0, 2\pi],$$

For every $\omega > 0$ and $\kappa > 0$ we have

$$\int_0^{2\pi} \frac{\partial_{N+1} \text{Kur}(s\mathbb{1}, s)}{\text{Kur}(s\mathbb{1}, s)} ds = - \int_0^{2\pi} \frac{\kappa}{\omega} ds = - \frac{2\pi\kappa}{\omega} < 0.$$

□

2 Dispersion curve

The strategy to prove the mains results is to use the comparison theorem of differentials equations. We assumed a priori that the distance between the oscillators is small and find some differential equation estimation to deduce that the distance between oscillators is bounded uniformly on time. We call the “upper-solution” *the dispersion curve*. We have the following lemma

Lemma 8. *Let $\Sigma = (\Sigma_1, \Sigma_2, \Sigma_3) \in \mathbb{R}_+^3 / \{(0, 0, 0)\}$. Let $P_1(a, b) = \Sigma_1 a + \Sigma_2 b^2$ a polynomial defined for all $(a, b) \in \mathbb{R} \times \mathbb{R}$ and let $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ a C^1 and 1-periodic function satisfying*

$$\int_0^1 \Lambda(s) ds < 0.$$

Then for all $(a, b) \in \mathbb{R}_+^ \times (0, \Sigma_3)$ the following differential equation*

$$\frac{d}{ds} z(s) = \frac{P_1(a, b)}{\Sigma_3 - b} + \Lambda(s)z(s), \quad (1)$$

admits a positive solution C^1 and 1-periodic solution that we note $\Delta_{a,b}(s)$. Further, there exists $D_{\Sigma, \Delta} \in (0, \Sigma_3)$ such that for all $D \in (0, D_{\Sigma, \Delta}]$ there exists $r > 0$ such that the solution $\Delta_r := \Delta_{r,D}$ satisfies

$$\max_{s \in [0, 1]} \Delta_r(s) \leq D.$$

Proof. Remark that for all $(a, b) \in \mathbb{R}_+^* \times (0, \Sigma_3)$, the differential equation (1) admit a positive C^2 and 1-periodic solution $\Delta_{a,b}(s)$ of the form

$$\Delta_{a,b}(s) = \frac{P_1(a, b)}{\Sigma_3 - b} \frac{\int_s^{1+s} \exp\left(\int_t^{1+s} \Lambda(v) dv\right) dt}{1 - \exp\left(\int_0^1 \Lambda(v) dv\right)}$$

Put

$$\lambda_1 = - \int_0^1 \Lambda(s) ds \quad \text{and} \quad \lambda_2 = \max_{0 \leq s, t \leq 1} \int_t^{1+s} \Lambda(v) dv.$$

$$\max_{s \in [0, 1]} \Delta_{a,b}(s) \leq \frac{P_1(a, b)}{\Sigma_3 - b} \frac{\exp(\lambda_2)}{1 - \exp(-\lambda_1)}.$$

To get $\max_{s \in [0, 1]} \Delta_r(s) \leq D$ it sufficient to choose r and D such that

$$\frac{P_1(r, D)}{\Sigma_3 - D} \frac{\exp(\lambda_2)}{1 - \exp(-\lambda_1)} = D, \quad (2)$$

which is satisfied for all $D \in (0, D_{\Sigma, \Lambda}]$ such that

$$D_{\Sigma, \Lambda} = \frac{\Sigma_3}{2} \frac{1 - \exp(-\lambda_1)}{1 - \exp(-\lambda_1) + \Sigma_2 \exp(\lambda_2)}.$$

where $r > 0$ is given by the following formula

$$r = \frac{D}{\Sigma_1} \left[\Sigma_3 \frac{1 - \exp(-\lambda_1)}{\exp(\lambda_2)} - \left[\frac{1 - \exp(-\lambda_1)}{\exp(\lambda_2)} + \Sigma_2 \right] D \right].$$

□

Definition 9. Let $D \in (0, D_{\Sigma, \Lambda}]$. We call *the dispersion curve associated to D* the solution

$$\Delta_r := \Delta_{r, D}(s),$$

of the differential equation (1) where r is defined by

$$r = \frac{D}{\Sigma_1} \left[\Sigma_3 \frac{1 - \exp(-\lambda_1)}{\exp(\lambda_2)} - \left[\frac{1 - \exp(-\lambda_1)}{\exp(\lambda_2)} + \Sigma_2 \right] D \right], \quad (3)$$

where

$$\lambda_1 = - \int_0^1 \Lambda(s) ds \quad \text{et} \quad \lambda_2 = \max_{0 \leq s, t \leq 1} \int_t^{1+s} \Lambda(v) dv.$$

Definition 10. Let $D \in (0, D_{\Sigma, \Lambda}]$. We call *the synchronization open set associated to D* and we note C_r the open set on \mathbb{R}^N defined by

$$C_r := \left\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N : \exists \nu_X \in \mathbb{R}, \quad \max_i |x_i - \nu_X| < \Delta_r(\nu_X) \right\}, \quad (4)$$

where Δ_r is the dispersion curve associated to D .

Remark 11. Remark that

$$D < D_{\Sigma, \Lambda} < \frac{\Sigma_3}{2\Sigma_2} \exp(-\lambda_2) \quad \text{and} \quad r < D \frac{\Sigma_3}{\Sigma_1} \exp(-\lambda_2).$$

3 Reduction of the system (P)

The goal of this Section is to prove that the perturbed system (P) in particular the periodic not-perturbed system (PNP) can be studied using a scalar periodic differential equation such as equation (1) of lemma 8. Now we define a new system

Definition 12. Let $X \in \mathbb{R}^N$ and let $\mu_0 \in \mathbb{R}$, we call the (NPS) *system associated to $\Phi^t(X)$* the not-perturbed following system

$$\dot{\mu}_X = F(\Phi^t(X), \mu_X), \quad t \in I_X, \quad (\text{NPS})$$

where $I_X = [t_0, T_X)$ is the maximal interval of the solution $X(t) := \Phi^t(X)$ of the system (P) of initial condition $\phi^{t_0}(X) = X$. We say that $\mu_X(t)$ is the solution of the system (NPS) associated to $\Phi^t(X)$ of initial condition $\mu_X(t_0) \in \mathbb{R}$.

We note

$$L := \|F\|_B + \|dF\|_B + \|d^2F\|_B, \quad \text{and} \quad \alpha := \min_{s \in [0,1]} F(s\mathbb{1}, s). \quad (5)$$

Let $X \in \mathbb{R}^N$ and let $\mu_X(t)$ be the solution of the system (NPS) associated to $\Phi^t(X)$ of initial condition $\mu_0 \in \mathbb{R}$. We also note $X := \Phi^t(X)$ and $\mu_X := \mu_X(t)$ without loss of generality. We consider the following quantities

$$\begin{aligned} \delta_{i,1}(X) &:= x_i - \mu_X, \quad \delta_{i,2}(X) := \mu_X - x_i, \\ \text{and} \quad \delta(X) &:= \max_{1 \leq i \leq N} |\delta_{i,1}(X)| = \max_{1 \leq i \leq N} |\delta_{i,2}(X)|. \end{aligned}$$

We have the next lemma

Proposition 13. *We consider the system (P). Suppose that the function F satisfies the hypothesis (H) and suppose that $\Phi^t(X)$ is defined for all $t \in [t_1, t_2]$. Let $D \in (0, \frac{\alpha}{L})$ and suppose that $\delta(X) < D$ for all $t \in [t_1, t_2]$, then*

$$\dot{\mu}_X > -LD + \alpha > 0, \quad \forall t \in [t_1, t_2].$$

In particular, $t \rightarrow \mu_X(t)$ is a diffeomorphism from $[t_1, t_2]$ to $[\mu_X(t_1), \mu_X(t_2)]$.

Proof. The strategy is to use the Mean value theorem. Since $\delta(X) < D$ we get, $|F(X, \mu_X) - F(\mu_X\mathbb{1}, \mu_X)| \leq \|dF\|_B D < LD$. Hence

$$\dot{\mu}_X = F(X, \mu_X) = [F(X, \mu_X) - F(\mu_X\mathbb{1}, \mu_X)] + F(\mu_X\mathbb{1}, \mu_X) > -LD + \alpha.$$

Thanks to hypothesis $0 < D < \frac{\alpha}{L}$ to get $\dot{\mu}_X(t) > -LD + \alpha > 0$ for all $t \in [t_1, t_2]$. \square

Proposition 14. *We consider the system (P). Suppose that F satisfies the hypothesis (H) and suppose that $\Phi^t(X)$ is defined for all $t \in [t_1, t_2]$. Let $D \in (0, \frac{\alpha}{L})$ and $r > 0$. Suppose that*

$$\|H\|_B < r, \quad \text{and} \quad \delta(X) < D, \quad \forall t \in [t_1, t_2].$$

Then for all $1 \leq i \leq N$, $k \in \{1, 2\}$ and $s \in [\mu_X(t_1), \mu_X(t_2)]$ we have

$$\frac{d}{ds} \delta_{i,k}^*(s) < \frac{1}{\alpha} \frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{\partial F_{N+1}(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} \delta_{i,k}^*(s), \quad (6)$$

where

$$\delta_{i,k}^*(s) := \delta_{i,k}(X(\mu_X^{-1}(s))) \quad \text{and} \quad X(\mu_X^{-1}(s)) = (x_1(\mu_X^{-1}(s)), \dots, x_N(\mu_X^{-1}(s))).$$

Proof. The strategy is to use several times the Taylor formula. Let $D \in (0, \frac{\alpha}{L})$ and suppose that $\delta(X) < D$ for all $t \in [t_1, t_2]$. Use the Taylor formula, there exists $c_i \in [x_i, \mu_X]$ such that for all $1 \leq i \leq N$

$$\begin{aligned} F(X, x_i) - F(X, \mu_X) &= \partial_{N+1}F(X, \mu_X)\delta_{i,1} + \frac{1}{2}\partial_{N+1}[\partial_{N+1}F(X, c_i)]\delta_{i,1}^2 \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,1} + \frac{1}{2}\|d\partial_{N+1}F\|_B D^2 \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,1} + \frac{1}{2}LD^2 \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,1} + LD^2. \end{aligned}$$

For $k = 2$ we also obtain

$$\begin{aligned} F(X, \mu_X) - F(X, x_i) &= -\partial_{N+1}F(X, \mu_X)\delta_{i,1} - \frac{1}{2}\partial_{N+1}[\partial_{N+1}F(X, c_i)]\delta_{i,1}^2 \\ &= \partial_{N+1}F(X, \mu_X)\delta_{i,2} - \frac{1}{2}\partial_{N+1}[\partial_{N+1}F(X, c_i)]\delta_{i,2}^2 \\ &< \partial_{N+1}F(X, \mu_X)\delta_{i,2} + LD^2. \end{aligned}$$

In other hand we have $\|H\|_B < r$. Use equations (P) and (NPS) we obtain for all $1 \leq i \leq N$ and $k \in \{1, 2\}$

$$\frac{d}{dt}\delta_{i,k} = H_i(X, x_i) + [F(X, x_i) - F(X, \mu_X)] < r + LD^2 + \partial_{N+1}F(X, \mu_X)\delta_{i,k}. \quad (7)$$

Use again the Taylor formula to get

$$\begin{aligned} \partial_{N+1}F(X, \mu_X)\delta_{i,k} &= [\partial_{N+1}F(X, \mu_X) - \partial_{N+1}F(\mu_X \mathbb{1}, \mu_X) + \partial_{N+1}F(\mu_X \mathbb{1}, \mu_X)]\delta_{i,k} \\ &< \|d\partial_{N+1}F\|_B D |\delta_{i,j}| + \partial_{N+1}F(\mu_X \mathbb{1}, \mu_X)\delta_{i,k} \\ &< LD^2 + \partial_{N+1}F(\mu_X \mathbb{1}, \mu_X)\delta_{i,k}. \end{aligned}$$

Equation (7) implies that for all $1 \leq i \leq N$ and $k \in \{1, 2\}$

$$\frac{d}{dt}\delta_{i,k} < r + 2LD^2 + \partial_{N+1}F(\mu_X \mathbb{1}, \mu_X)\delta_{i,k}.$$

Thanks to proposition 13, $\dot{\mu}_X > \alpha - LD$. We consider the change of variable : $t \rightarrow s := \mu_X(t)$ for $t \in [t_1, t_2]$. Put $\delta_{i,k}^*(s) := \delta_{i,k}(X(\mu_X^{-1}(s)))$ and $X(\mu_X^{-1}(s)) = (x_1(\mu_X^{-1}(s)), \dots, x_N(\mu_X^{-1}(s)))$. We deduce that for all

$$s \in [\mu_X(t_1), \mu_X(t_2)]$$

$$\begin{aligned} \frac{d}{dt}\delta_{i,k}(X) &= \frac{d}{ds}\delta_{i,k}^*(s)\frac{d}{dt}\mu_X(t) < r + 2LD^2 + \partial_{N+1}F(s\mathbb{1}, s)\delta_{i,k}^*(s) \\ \frac{d}{ds}\delta_{i,k}^*(s) &= \frac{r + 2LD^2}{\dot{\mu}_X} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{\dot{\mu}_X}\delta_{i,k}^*(s) < \frac{r + 2LD^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{\dot{\mu}_X}\delta_{i,k}^*(s) \\ &= \frac{r + 2LD^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\frac{F(s\mathbb{1}, s)}{\dot{\mu}_X}\delta_{i,k}^*(s). \end{aligned}$$

Use the Mean value theorem and the change of variable $t \rightarrow s := \mu_X(t)$ we get

$$|F(\mu_X\mathbb{1}, \mu_X) - \dot{\mu}_X| = |F(\mu_X\mathbb{1}, \mu_X) - F(X, \mu_X)| < \|dF\|_B D < LD,$$

which is equivalent to

$$\frac{F(s\mathbb{1}, s)}{\dot{\mu}_X} = 1 + \theta(s), \quad |\theta(s)| < \frac{LD}{\alpha - LD}, \quad \forall s \in [\mu_X(t_1), \mu_X(t_2)].$$

Finlay, since $|\frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}| < \frac{L}{\alpha}$ and since $|\delta_{i,k}(t)| \leq \delta(X) < D$ for all $t \in [t_1, t_2]$ we obtain for all $s \in [\mu_X(t_1), \mu_X(t_2)]$

$$\begin{aligned} \frac{d}{ds}\delta_{i,k}^*(s) &< \frac{r + 2LD^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}[1 + \theta(s)]\delta_{i,k}^*(s) \\ &< \frac{r + 2LD^2}{\alpha - LD} + \frac{L^2}{\alpha} \frac{D^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\delta_{i,k}^*(s) \\ &= \frac{1}{\alpha} \frac{\alpha(r + 2LD^2) + L^2 D^2}{\alpha - LD} + \frac{\partial_{N+1}F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}\delta_{i,k}^*(s). \end{aligned}$$

□

The previous lemma allows us to prove the existence of an open Φ^t -invariant set. We have the following proposition

Proposition 15. *Let F be a function satisfying hypotheses (H) and (H_*) . Then there exists $D_* \in (0, 1)$ such that for all $D \in (0, D_*]$, there exists $r > 0$ and an open set C_r (as in definition 10), such that for any function H satisfying $\|H\|_B < r$ we have*

$$\forall X \in C_r : \Phi^t(X) \in C_r, \quad \forall t \in I_X.$$

Proof. Use equation (5) and hypothesis (H)

$$\max\left\{\int_{t_1}^{t_2} \frac{\partial_{N+1} F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} ds : 0 \leq t_1 \leq t_2 \leq 1\right\} \leq \frac{L}{\alpha}.$$

Let $D_{\Sigma, \Lambda}$ the constant defined by lemma 8 such that Σ et λ are defined by

$$\Sigma = \left(\frac{1}{L}, 2 + \frac{L}{\alpha}, \frac{\alpha}{L}\right), \quad \text{and} \quad \Lambda(s) = \frac{\partial_{N+1} F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)}.$$

Put $D_* := D_{\Sigma, \Lambda}$. Let $D \in (0, D_*]$ and the dispersion function Δ_r associated to D (See definition 9). The dispersion curve Δ_r is solution of the periodic scalar differential equation

$$\frac{d}{ds} \Delta_r(s) = \frac{1}{\alpha} \frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{F_{N+1}(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} \Delta_r(s),$$

and satisfies the folioing estimation

$$\max_{s \in [0, 1]} \Delta_r(s) \leq D.$$

Let C_r be the synchronization open set associated to D , as defined in definition 10.

For any function H satisfying $\|H\|_B < r$, where r is given by formula (3), let $X(t) = (x_1(t), \dots, x_N(t)) := \Phi^t(X)$ be the solution of the system (P) of initial condition $X = (x_1, \dots, x_N) \in C_r$. There exists $\nu_X \in \mathbb{R}$ such that $\max_{1 \leq i \leq N} |x_i - \nu_X| < \Delta_r(\nu_X) \leq D$. Let $\mu_X(t)$ be the solution of the system (NPS) associated to $X(t)$ of initial condition $\mu_X(t_0) = \nu_X$, then $\delta(X) < \Delta_r(\mu_X(t_0))$. Let

$$T^* := \sup\{t \in I_X : \forall t_0 < s < t, \max_i |x_i(s) - \mu_X(s)| < \Delta_r(\mu_X(s))\}.$$

By continuity we have $t_0 \neq T^*$. The proposition is proved if we shows that $T^* = \sup\{t \in I_X\}$. By contradiction, suppose that $T^* \in I_X$. Using the change of variable $s = \mu_X(t)$ the proposition 14 implies that for all $s \in [\nu_X, \mu_X^* := \mu_X(T^*)]$

$$\frac{d}{ds} \delta_{i,k}^*(s) < \frac{1}{\alpha} \frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{F_{N+1}(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} \delta_{i,k}^*(s), \quad \forall s \in [\mu_X, \mu_X^*].$$

Hence there exists $1 \leq i_0 \leq N$ and $k \in \{1, 2\}$ such that $|\delta_{i_0, k_0}^*(\mu_X^*)| = \Delta_r(\mu_X^*)$. Suppose that $\delta_{i_0, k_0}^*(\mu_X^*) = \Delta_r(\mu_X^*)$ without loss

of generality. We get

$$\begin{aligned} \frac{d}{ds} \delta_{i_0, k_0}^*(\mu_X^*) &< \frac{1}{\alpha} \frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{F_{N+1}(\mu_X^* \mathbb{1}, \mu_X^*)}{F(\mu_X^* \mathbb{1}, \mu_X^*)} \delta_{i_0, k_0}^*(\mu_X^*) \\ &= \frac{1}{\alpha} \frac{\alpha r + LD^2(L + 2\alpha)}{\alpha - LD} + \frac{F_{N+1}(\mu_X^* \mathbb{1}, \mu_X^*)}{F(\mu_X^* \mathbb{1}, \mu_X^*)} \Delta_r(\mu_X^*) = \frac{d}{ds} \Delta_r(\mu_X^*). \end{aligned}$$

There exists $s < \mu_X^*$ close enough to μ_X^* such that $\delta_{i_0, k_0}^*(s) > \Delta_r(s)$ or in other words there exists $t < T^*$ close enough to T^* such that $\delta_{i_0, k_0}(t) > \Delta_r(\mu_X(t))$. We have obtained a contradiction. \square

4 Proof of main result I : Existence of solution and the synchronized state

Now we prove the main result **I** (which shows the existence of solution for all $t \geq t_0$ and the existence of a synchronized state) by proving the next theorem

Theorem 16. *Let F be a function satisfying the hypotheses (H) and (H_*) . Then there exists $D_* \in (0, 1)$ such that for all $D \in (0, D_*]$, there exists $r > 0$ and an synchronization open set C_r (as defined in the definition 10), such that for any function H satisfying $\|H\|_B < r$, and for all $X \in C_r$ we have $I_X = [t_0, +\infty[$. Further C_r is positive Φ^t -invariant and*

$$\forall X \in C_r, \exists \nu_X \in \mathbb{R} : |\Phi_i^t(X) - \mu_X(t)| < D, \forall i = 1, \dots, N, \forall t \geq t_0,$$

where $\mu_X(t)$ is the solution of the system (NPS) associated to $\Phi^t(X)$ of initial condition $\mu_X(t_0) = \nu_X$.

Proof. Thanks to proposition 15, it is sufficient to prove that $I_X = [t_0, +\infty[$. By contradiction suppose that there exists $t_0 < t_X < +\infty$ such that the solution $X(t)$ is defined only on $I_X = [t_0, t_X[$. Then $\lim_{t \rightarrow t_X} \|\Phi^t(t)\| = +\infty$. Proposition 15 implies that

$$|\Phi_i^t(X) - \Phi_j^t(X)| < D, \forall 1 \leq i, j \leq N, \forall t_X > t \geq t_0,$$

For all $i = 1, \dots, N$,

$$\alpha - LD - r < \frac{d}{dt} x_i < \max_{s \in [0, 1]} F(s \mathbb{1}, s) + LD + r, \forall t_X > t \geq t_0.$$

Hence $\|\Phi^t(X)\| < +\infty$ for all $t \in [t_0, t_X]$, in particular $\lim_{t \rightarrow t_X} \|\Phi^t(t)\| < +\infty$. We have obtained a contradiction. \square

5 Proof of main result II : Periodic locked solution

We use the fixed point theorem to prove the existence of periodic locked state as follow

Lemma 17. *Let F be a function satisfying the hypotheses (H) and (H_*) . For any $\mathbb{1}$ -periodic C^1 function H satisfying $\|H\|_B < r$ let C_r the synchronization Φ^t -invariant open set given by theorem 16. Let the set Σ defined by*

$$\Sigma = \{X \in \mathbb{R}^N, \max_i |x_i| < \Delta_r(0)\} \subset C_r.$$

Then there exists a C^1 function $P : \Sigma \rightarrow \Sigma$ (the Poincaré map) and a C^1 function $\theta : \Sigma \rightarrow \mathbb{R}^+$ (the return time map) such that

$$\begin{aligned} \Phi^{t_0+\theta(X)}(X) &= P(X) + \mathbb{1}, \quad \mathbb{1} = (1, \dots, 1) \in \mathbb{R}^N, \\ \frac{1}{L} &< \theta(X) < \frac{2}{\alpha}. \end{aligned}$$

Proof. Let $X \in \Sigma \subset C_r$. Let $\mu_X(t)$ be the solution of the system (NPS) associated to $X(t)$ of initial condition $\mu_X(t_0) = 0$. Let τ_X be the inverse function of the function $\mu_X := \mu_X(t)$. By the proposition 13 and the theorem 16 we obtain

$$\alpha - LD < \dot{\mu}_X(t) < L.$$

Remark (11) in the Section 2 shows that for all $D < \frac{\alpha}{2L}$ we have

$$\frac{\alpha}{2} < \dot{\mu}_X(t) < L.$$

Let $\theta(X) := \tau_X(1) - t_0$. Then $\int_{t_0}^{\tau_X(1)} \dot{\mu}_X(t) dt = 1$ which implies the second estimation of lemma. Recall that $\max_{1 \leq i \leq N} |\Phi_i^t(X) - \mu_X(t)| < \Delta(\mu_X(t)) < D$ for all $t \geq t_0$. Put $P(X) := \Phi^{t_0+\theta(X)}(X) - \mathbb{1}$, $P = (P_1, \dots, P_N)$. Since $\mu_X(t_0 + \theta(X)) = 1$ then

$$\max_{1 \leq i \leq N} |P_i(X)| = \max_{1 \leq i \leq N} |\Phi_i^{t_0+\theta(X)}(X) - 1| < \Delta(1) = \Delta(0).$$

We have shown that P is a map from Σ into itself. □

Corollary 18. *The Poincaré map P defined in lemma 17 admits a fixed point $X_* \in \Sigma$.*

Proof. $\bar{\Sigma}$ is compact and convex; $P : \bar{\Sigma} \rightarrow \bar{\Sigma}$ is continuous. By Brouwer fixed point theorem, P admits a fixed point in $X_* \in \bar{\Sigma}$. We claim that $X_* \notin \partial\bar{\Sigma}$. Suppose by contradiction $X_* \in \partial\bar{\Sigma}$. There exists $1 \leq i_0 \leq N$, such that $|x_{i_0,*}| = \Delta_r(0)$; Put

$$X_*(t) = \Phi^t(X_*), \quad X_*(t) = (x_{1,*}(t), \dots, x_{N,*}(t)).$$

Let $\mu_{X_*}(t)$ the solution of the system (NPS) associated to the solution $X_*(t)$ of initial condition $\mu_{X_*}(t_0) = 0$. We note

$$\delta_{i,1}(X_*(t)) = x_{i,*}(t) - \mu_{X_*}(t), \quad \text{and} \quad \delta_{i,2}(X_*(t)) = \mu_{X_*}(t) - x_{i,*}(t).$$

There exists $1 \leq i \leq N$, $k \in \{1, 2\}$, and $t' > t_0$ close to t_0 such that $\delta_{i_0,k}(X_*(t)) < \Delta_r(\mu_{X_*}(t))$ for all $t' > t > t_0$. By repeating this argument for every $1 \leq i_1 \leq N$ and $k \in \{1, 2\}$ satisfying the equality $\delta_{i_1,k}(X_*(t)) = \Delta_r(\mu_{X_*}(t))$ we obtain for some $t^* > t_0$, $\delta(X_*(t^*)) < \Delta_r(\mu_{X_*}(t^*))$. But 16 implies that $\max_{1 \leq i \leq N} |x_{i,*}(t) - \mu_{X_*}(t)| < \Delta_r(\mu_{X_*}(t))$ for all $t > t^*$. We have obtained a contradiction with the fact that

$$\begin{aligned} \max_{1 \leq i \leq N} |\Phi_i^{t_0+\theta(X_*)}(X_*) - \mu_{X_*}(t_0 + \theta(X_*))| &= \max_{1 \leq i \leq N} |x_{i,*} + 1 - 1| \\ &= \Delta_r(0) = \Delta_r(\mu_{X_*}(t_0 + \theta(X_*))), \end{aligned}$$

knowing that $\mu_{X_*}(t_0 + \theta(X_*)) = 1$ and $\Phi^{t_0+\theta(X_*)}(X_*) = X_* + \mathbb{1}$. \square

The main result **II** which show the existence of periodic locked solution is a consequence of previous corollary.

Proof of the main result II. Corollary 18 implies the existence of a fixed point $X_* \in C_r$ and return time $\theta_* > 0$ such that

$$\Phi^{t_0+\theta_*}(X_*) = X_* + \mathbb{1}.$$

Thanks to periodicity and uniqueness of solution of differential equation, we obtain

$$\Phi^{\theta_*+t}(X_*) = \Phi^t(X_*) + \mathbb{1}, \quad \forall t \geq t_0.$$

Let $\Psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ the function defined by

$$\Psi(s) := \Phi^s(X_*) - \frac{s}{\theta_*} \mathbb{1} = (\Psi_1(s), \dots, \Psi_N(s)), \quad \forall s \geq t_0.$$

The theorem is proved if we show that Ψ_i are θ_* -periodic. We have

$$\Psi(s + \theta_*) = \Phi^{s+\theta_*}(X_*) - \frac{s + \theta_*}{\theta_*} \mathbb{1} = \Phi^s(X_*) + \mathbb{1} - \frac{s + \theta_*}{\theta_*} \mathbb{1} = \Psi(s).$$

Lemma 17 implies that θ_* is uniformly bounded. \square

6 Numerical Example : Linear mean field system

We see in [8] that the hypothesis (H_*) called the “synchronization hypothesis” seems to be a bifurcation criterion for the existence of synchronization domain for the Winfree model. When the Winfree model is not perturbed the N oscillators are always synchronized, independently of their initial conditions, thanks to the uniqueness property of periodic solutions of an ODE, since the system is periodic relative to each variable. Now we show an example of a \mathbb{T} -periodic system but not periodic relative to each variable. Numerically the hypothesis (H_*) seems again to be a bifurcation criterion for the existence of synchronization also when the system is not perturbed.

We consider the linear mean field system

$$\dot{x}_i = \omega_i + \beta \frac{1}{N} \sum_{j=1}^N (x_i - x_j), t \geq t_0, \quad (8)$$

where $N \geq 2$, $(\omega_i)_i$ are the natural frequencies on $[1 - \gamma, 1 + \gamma]$ and $\gamma \in [0, 1[$ the spectrum width. β is a parameter in $[-0.5, 0.5]$. The model (8) satisfies the hypothesis (H) . In fact we can take the function F as

$$F(Y, z) = 1 + \beta \frac{1}{N} \sum_{j=1}^N (z - y_j), \quad \forall Y = (y_1, \dots, y_N) \in \mathbb{R}^N, \quad z \in \mathbb{R},$$

which is C^∞ and $F(s\mathbb{1}, s) = 1$ is a constant (periodic) function such that $\min_{s \in \mathbb{R}} F(s\mathbb{1}, s) = 1 > 0$. Remark that the hypothesis (H_*) is satisfied if and only if $\beta < 0$. Unlike the case of the Winfree model in [8] we observe in Figure 1 that the dispersion is not bounded when (H_*) is not satisfied also when $\gamma = 0$.

7 Conclusion and open problem

We have generalized the result obtained in [8] to a class of abstract mean field models. We have proved for a small perturbations the existence of solution and the existence of the synchronized solution. When the system is periodic we have proved in addition the existence periodic locked state. In the numerical results of [8] and the Section 6 of this article the hypothesis (H_*) seems to be a bifurcation criterion for the existence of synchronization, we conjecture that the hypothesis (H_*) is necessary to get the synchronization of the oscillators of the mean field systems defined in this article. More

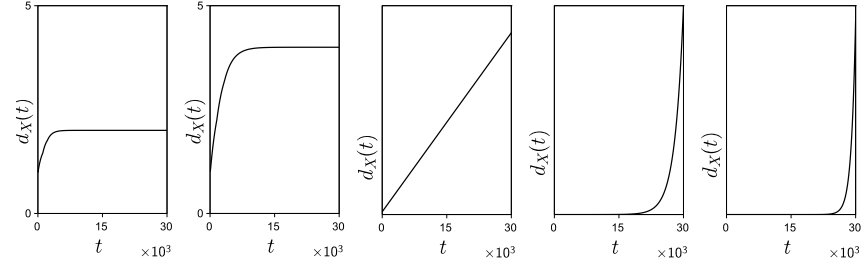
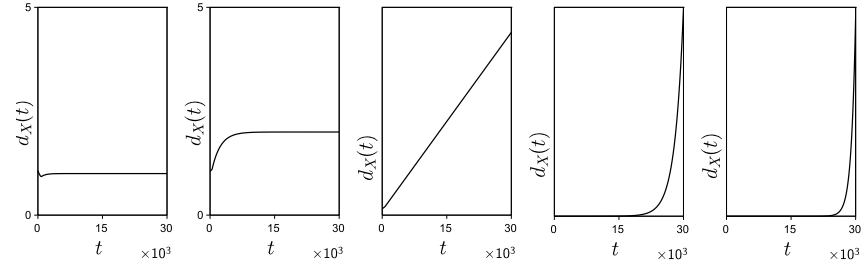
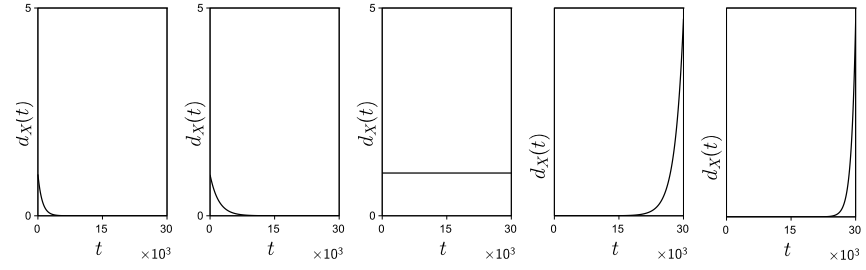
(a) $\gamma_2 = 0.0412$ (b) $\gamma_1 = 0.011$ (c) $\gamma_0 = 0$

Figure 1: We choose on the model 8 a random initial conditions X in $[-1, 1]$. We fix $N = 100$ oscillators and we choose a uniform distribution of the natural infrequencies ω_i in $[1 - \gamma, 1 + \gamma]$. We plot the dispersion $d_X(t) := \delta(X)$ for $t \in [0, \times 10^4]$. We choose vertically from bottom to top, $\gamma_0 = 0$, $\gamma_1 = 0.001$ and $\gamma_2 = 0.002$, horizontally from left to right, $\beta = -0.001$, $\beta = -0.0005$, $\beta = 0$, $\beta = 0.0005$ and $\beta = 0.001$.

precisely : when

$$\begin{aligned} \int_0^1 \frac{\partial_{N+1} F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} ds < 0 & \quad \text{the oscillators are synchronized for small perturbations.} \\ \int_0^1 \frac{\partial_{N+1} F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} ds > 0 & \quad \text{the oscillators are desynchronized.} \\ \int_0^1 \frac{\partial_{N+1} F(s\mathbb{1}, s)}{F(s\mathbb{1}, s)} ds = 0 & \quad \text{the test is inconclusive.} \end{aligned}$$

References

- [1] J.T. Ariaratnam, and S.H. Strogatz, Phase Diagram for the Winfree Model of Coupled Nonlinear Oscillators, *Phys. Rev. Lett.* 86, 4278 (2001).
- [2] L. Basnarkov, and V. Urumov, Critical exponents of the transition from incoherence to partial oscillation death in the Winfree model, *J. Stat. Mech.* P10014 (2009)
- [3] D. Cumin. and C.P. Unsworth, Generalising the Kuramoto model for the study of neuronal synchronization in the brain. *Physica D: Nonlinear Phenomena*, **2**, 226, 181–196. (2007).
- [4] G. B. Ermentrout and M. Pascal and B. S. Gutkin, The Effects of Spike Frequency Adaptation and Negative Feedback on the Synchronization of Neural Oscillators. *Neural Computation*, **6**, 13, 1285-1310.(2001)
- [5] F. Giannuzzi, D. Marinazzo, G. Nardulli, M. Pellicoro, and S. Stramaglia, Phase diagram of a generalized Winfree model, *Phys. Rev. E* 75, 051104.(2007)
- [6] S.-Y. Ha, J. Park, and S. W. Ryoo, Emergence of phase-locked states for the Winfree model in a large coupling regime . *Discrete and Continuous Dynamical Systems, Series A*, 35 , no. 8, 3417-3436 (2015)
- [7] S, Louca, and F. M. Atay, Spatially structured networks of pulse-coupled phase oscillators on metric spaces *AIMS Journal of Discrete and Continuous Dynamical Systems - A*, vol. 34 (2014)

- [8] W. Oukil, A. Kessi and Ph. Thieullen, Synchronization hypothesis in the Winfree model. Dynamical System, doi = 10.1080/14689367.2016.1227303 (2016)
- [9] D. Pazó and E. Montbrió, Low-dimensional dynamics of populations of pulse-coupled oscillators, Phys. Rev. X 4 011009 (2014).
- [10] O.V. Popovych, YL Maistrenko, PA Tass. Phase chaos in coupled oscillators. Physical Review E 71, 065201 (R), (2005)
- [11] Y. Kuramoto, International Symposium on Mathematical Problems in Theoretical Physics. Lecture Notes in Physics, Springer, New York, **39** 420-20.(1975)
- [12] D.D. Quinn, R. H. Rand and S.H. Strogatz, Singular unlocking transition in the Winfree model of coupled oscillators. Physical Review E 75, 036218 (2007)
- [13] A. T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators *J. Theor. Biol.* **16** 15-42.(1967)